Lesson 10

Statistics III

Todays topics

- **interval estimation (区間推定)**
- **hypothesis testing (仮説検定)**
  - **t-test**
  - **χ^2-test**

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1. Interval estimation

<u>Statistical Inference (統計的推定)</u>

- ✓ point estimation (点推定)
  - ➢ consistent estimation (一致推定)
  - ➢ unbiased estimation (不偏推定)
  - ➢ maximum likelihood (最尤推定)
- ✓ interval estimation (区間推定)

Statistical inference

Example 1

A clerk says “our eggs are big. 70[g] in average.”

You bought 6 eggs in a shop.

	1	2	3	4	5	6
weight[g]	64.3	70.4	63.2	67.8	71.3	60.8

How large are eggs sold in this shop?

$\bar{X}$ = 66.3[g], s^2 = 17.584[g^2]

Is the clerk honest?

Central Limit Theorem (中心極限定理)

Def.

A series $\{Y_n\}$ w/ distribution functions $\{F_n\}$

converges Y in distribution (Yに分布収束する), if

$\lim_{n\to\infty} F_n = F$ where F is the distr. func. of Y.

Thm. Central limit theorem

Suppose $X_1, \dots, X_n$ are i.i.d., w/ expectation μ, and variance σ^2,

then $Z_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$ converges to $\mathrm{N}(0,1)$ in distribution.

i.e., $\lim_{n\to\infty} \Pr[Z_n < z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x$

$$Z_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma} = \frac{n}{\sigma\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{n} = \frac{1}{\sqrt{\frac{\sigma}{n}}} (\overline{X} - \mu)$$

Statistical inference

Example 1

A clerk says "our eggs are big. 70[g] in average."

$\bar{X}$ = 66.3[g], s^2 = 17.584[g^2] for 6 eggs.

Suppose σ^2=18.0 for simplicity.

Let z* (>0) satisfy

$$\Pr\left[-z^* \le \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \le z^*\right] \ge 0.95$$

"two-sided 95% confidence interval"

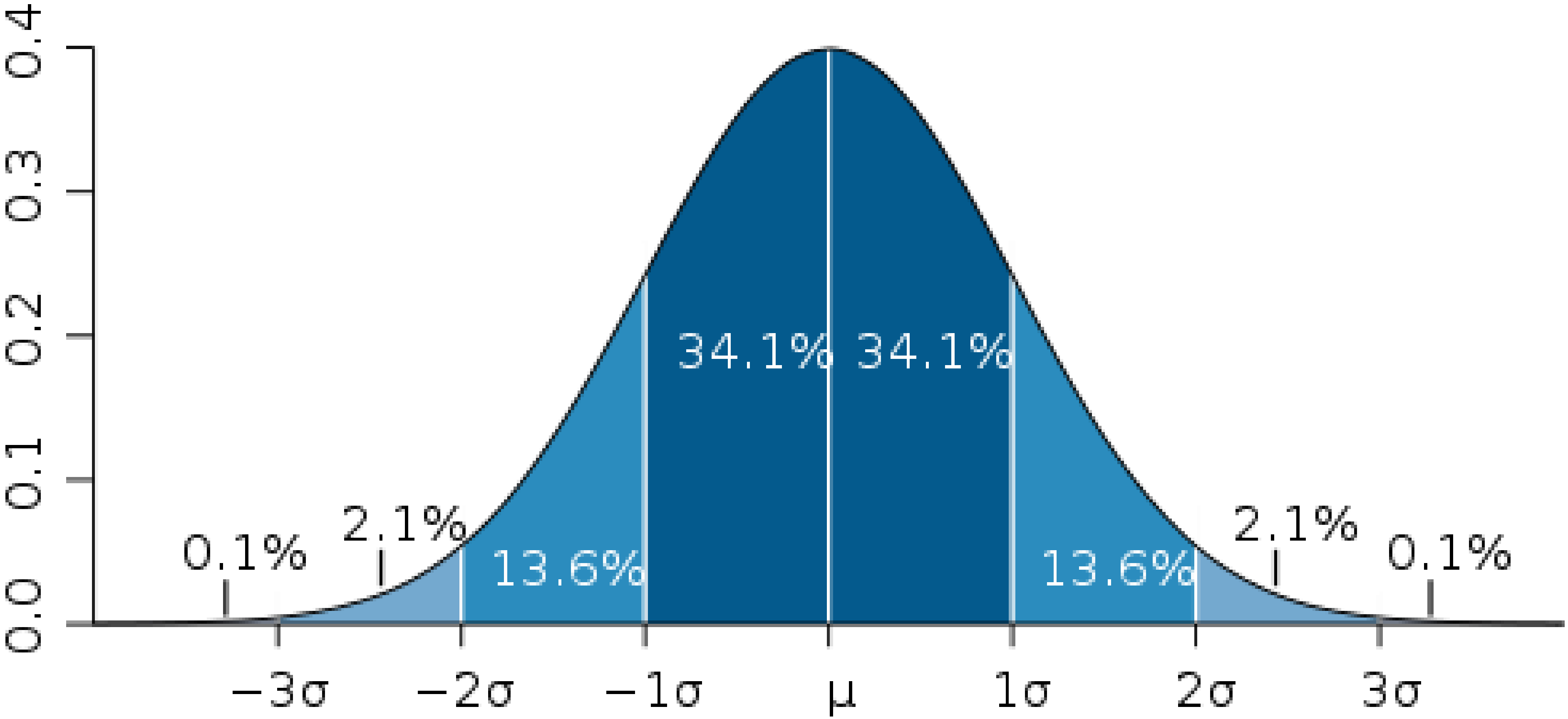
両側95%信頼区間

Since central limit theorem,

$$\Pr\left[-z^* \le \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \le z^*\right] = \int_{-z^*}^{z^*} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}x^2\right) \mathrm{d}x$$

… and we see that z* = 1.960 (see normal distribution table).

Normal distribution



Wikipedia: Standard normal table

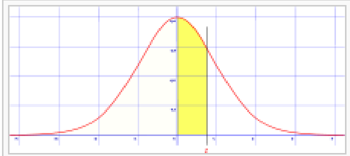
http://en.wikipedia.org/wiki/Normal_distribution

Standard normal table (標準正規分布表)

Cumulative from mean (0 to Z)

[edit]

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00000	0.00399	0.00798	0.01197	0.01595	0.01994	0.02392	0.02790	0.03188	0.03586
0.1	0.03983	0.04380	0.04776	0.05172	0.05567	0.05962	0.06356	0.06749	0.07142	0.07535
0.2	0.07926	0.08317	0.08706	0.09095	0.09483	0.09871	0.10257	0.10642	0.11026	0.11409
0.3	0.11791	0.12172	0.12552	0.12930	0.13307	0.13683	0.14058	0.14431	0.14803	0.15173
0.4	0.15542	0.15910	0.16276	0.16640	0.17003	0.17364	0.17724	0.18082	0.18439	0.18793
0.5	0.19146	0.19497	0.19847	0.20194	0.20540	0.20884	0.21226	0.21566	0.21904	0.22240
0.6	0.22575	0.22907	0.23237	0.23565	0.23891	0.24215	0.24537	0.24857	0.25175	0.25490
0.7	0.25804	0.26115	0.26424	0.26730	0.27035	0.27337	0.27637	0.27935	0.28230	0.28524
0.8	0.28814	0.29103	0.29389	0.29673	0.29955	0.30234	0.30511	0.30785	0.31057	0.31327
0.9	0.31594	0.31859	0.32121	0.32381	0.32639	0.32894	0.33147	0.33398	0.33646	0.33891
1.0	0.34134	0.34375	0.34614	0.34849	0.35083	0.35314	0.35543	0.35769	0.35993	0.36214
1.1	0.36433	0.36650	0.36864	0.37076	0.37286	0.37493	0.37698	0.37900	0.38100	0.38298
1.2	0.38493	0.38686	0.38877	0.39065	0.39251	0.39435	0.39617	0.39796	0.39973	0.40147
1.3	0.40320	0.40490	0.40658	0.40824	0.40988	0.41149	0.41308	0.41466	0.41621	0.41774
1.4	0.41924	0.42073	0.42220	0.42364	0.42507	0.42647	0.42785	0.42922	0.43056	0.43189
1.5	0.43319	0.43448	0.43574	0.43699	0.43822	0.43943	0.44062	0.44179	0.44295	0.44408
1.6	0.44520	0.44630	0.44738	0.44845	0.44950	0.45053	0.45154	0.45254	0.45352	0.45449
1.7	0.45543	0.45637	0.45728	0.45818	0.45907	0.45994	0.46080	0.46164	0.46246	0.46327
1.8	0.46407	0.46485	0.46562	0.46638	0.46712	0.46784	0.46856	0.46926	0.46995	0.47062
1.9	0.47128	0.47193	0.47257	0.47320	0.47381	0.47441	0.47500	0.47558	0.47615	0.47670
2.0	0.47725	0.47778	0.47831	0.47882	0.47932	0.47982	0.48030	0.48077	0.48124	0.48169
2.1	0.48214	0.48257	0.48300	0.48341	0.48382	0.48422	0.48461	0.48500	0.48537	0.48574
2.2	0.48610	0.48645	0.48679	0.48713	0.48745	0.48778	0.48809	0.48840	0.48870	0.48899
2.3	0.48928	0.48956	0.48983	0.49010	0.49036	0.49061	0.49086	0.49111	0.49134	0.49158
2.4	0.49180	0.49202	0.49224	0.49245	0.49266	0.49286	0.49305	0.49324	0.49343	0.49361
2.5	0.49[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2.6	0.49[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]



This table gives a probability that a statistic is between 0 (mean) and Z.

Wikipedia: Standard normal table

http://en.wikipedia.org/wiki/Standard_normal_table

Statistical inference

Example 1

A clerk says "our eggs are big. 70[g] in average."

$\bar{X}$ = 66.3[g], s^2 = 17.584[g^2] for 6 eggs.

Suppose σ^2=18.0 for simplicity.

$$\Pr\left[-z^* \le \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \le z^*\right] = \Pr\left[-z^*\frac{\sigma}{\sqrt{n}} \le \bar{X}-\mu \le z^*\frac{\sigma}{\sqrt{n}}\right]$$

$$= \Pr\left[-\bar{X}-z^*\frac{\sigma}{\sqrt{n}} \le -\mu \le -\bar{X}+z^*\frac{\sigma}{\sqrt{n}}\right]$$

$$= \Pr\left[\bar{X}+z^*\frac{\sigma}{\sqrt{n}} \ge \mu \ge \bar{X}-z^*\frac{\sigma}{\sqrt{n}}\right]$$

$\bar{X} = 66.3$[g]

$z^* = 1.960$

$\sigma^2 = 18.0$

$n = 6$

$$= \Pr\left[66.3+1.960\frac{\sqrt{18}}{\sqrt{6}} \ge \mu \ge 66.3-1.960\frac{\sqrt{18}}{\sqrt{6}}\right]$$

$$= \Pr[69.69 \ge \mu \ge 62.91]$$

Todays topics

- interval estimation (区間推定)
- hypothesis testing (仮説検定)
  - **t-test**
  - **χ^2-test**

2. hypothesis testing (仮説検定)

Hypothesis testing (仮説検定)

Terminology

- null hypothesis (帰無仮説)
- alternative hypothesis (対立仮説)

Idea

Pr[null hypo is true] $< \alpha$

$\Rightarrow$ reject the null hypothesis with significant level α

(有意水準αで帰無仮説を棄却する)

Pr[null hypo is true] $\geq \alpha$

$\Rightarrow$ fail to reject the null hypothesis with significant level α

(有意水準αで帰無仮説を棄却しない)

Statistical inference

Example 1

A clerk says “our eggs are big. 70[g] in average.”

You bought 6 eggs in a shop.

	1	2	3	4	5	6
weight[g]	64.3	70.4	63.2	67.8	71.3	60.8

How large are eggs sold in this shop?

$\bar{X}$ = 66.3[g], s^2 = 17.584[g^2]

Is the clerk honest?

Statistical inference

Example 1

A clerk says “our eggs are big. 70[g] in average.”

$\bar{X}$ = 66.3[g], s^2 = 17.584[g^2] for 6 eggs.

Let assume μ = 70.0

Suppose σ^2=18.0 for simplicity.

$$\Pr\left[-z^* \le \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \le z^*\right] = \Pr\left[-z^*\frac{\sigma}{\sqrt{n}} \le \bar{X}-\mu \le z^*\frac{\sigma}{\sqrt{n}}\right]$$

$\mu = 70$

$z^* = 1.960$

$\sigma^2 = 18.0$

$n = 6$

$\bar{X} = 66.3$[g]

→

$$= \Pr\left[\mu - z^*\frac{\sigma}{\sqrt{n}} \le \bar{X} \le \mu + z^*\frac{\sigma}{\sqrt{n}}\right]$$

$$= \Pr\left[70 - 1.960\frac{\sqrt{18}}{\sqrt{6}} \le \bar{X} \le 70 + 1.960\frac{\sqrt{18}}{\sqrt{6}}\right]$$

$$= \Pr\left[66.6 \le \bar{X} \le 73.4\right]$$

It rejects the null hypothesis μ = 70.0 with significant level 5%

(帰無仮説 μ = 70.0 は有意水準5%で棄却される。)

Todays topics

- interval estimation (区間推定)
- hypothesis testing (仮説検定)
  - **t-test**
  - **χ^2-test**

2. t distribution, χ^2 distribution

Student's t-statistics (スチューデントのt統計量)

Example 1

A clerk says "our eggs are big. 70[g] in average."

$\bar{X}$ = 66.3[g], s^2 = 17.584[g^2] for 6 eggs.

Let assume μ = 70.0 ~~Suppose σ^2=18.0 for simplicity.~~

Let $t := \frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}}$, ← $Z_n := \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$ in Cent. limit. Thm.

where $s^2 := \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1}$ (unbiased estimator of σ^2).

Question

Does t follow N(0,1), in a similar way as Z?

Student's t-statistics (スチューデントのt統計量)

Question

Does t follow $\mathrm{N}(0,1)$, in a similar way as z?

Let $t = \frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}}$ and $Z = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$

where $s^2 := \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1}$ (unbiased estimator of σ^2).

$$t = Z\frac{\sigma}{s} = \frac{Z}{\sqrt{\frac{s^2}{\sigma^2}}} = \frac{Z}{\sqrt{\frac{1}{\sigma^2}\cdot\frac{\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2}{n-1}}} = \frac{Z}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2}}$$

t-distribution and χ^2distribution

$$t = Z\frac{\sigma}{s} = \frac{Z}{\sqrt{\frac{s^2}{\sigma^2}}} = \frac{Z}{\sqrt{\frac{1}{\sigma^2}\cdot\frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{n-1}}} = \frac{Z}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2}}$$

Prop. 1.

$\sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2$ follows the χ^2-distribution with $n-1$ degrees.

χ^2-distribution with n degrees of freedom

Prop. 2.

$X_1, \ldots, X_n \sim \mathrm{N}(0,1)$, independently.

Let $Y := X_1^2 + \cdots + X_n^2$, then Y follows $\mathrm{Ga}\left(\frac{1}{2}, \frac{n}{2}\right)$.

Idea of Prop. 1

$$\sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2 = \sum_{i=1}^{n}\left(\frac{(X_i-\mu)-(\overline{X}-\mu)}{\sigma}\right)^2$$
$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}\left((X_i-\mu)^2 - 2(X_i-\mu)(\overline{X}-\mu) + (\overline{X}-\mu)^2\right)$$
$$= \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 - 2(\overline{X}-\mu)\frac{\sum_{i=1}^{n}(X_i-\mu)}{\sigma^2} + n\left(\frac{\overline{X}-\mu}{\sigma}\right)^2$$
$$= \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 - 2n\left(\frac{\overline{X}-\mu}{\sigma}\right)^2 + n\left(\frac{\overline{X}-\mu}{\sigma}\right)^2$$
$$= \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 - n\left(\frac{\overline{X}-\mu}{\sigma}\right)^2$$
$$= \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 - \left(\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2$$

<u>Rem.</u> if $X \sim \mathrm{N}(\mu,\sigma^2)$ then
$$\frac{X-\mu}{\sigma} \sim \mathrm{N}(0,1)$$

<u>Rem.</u> if $X \sim \mathrm{N}(\mu,\sigma^2)$ then
$$\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{\sqrt{n}}\right)$$

t-distribution and χ^2distribution [William Gosset]

$$t = Z\frac{\sigma}{s} = \frac{Z}{\sqrt{\frac{s^2}{\sigma^2}}} = \frac{Z}{\sqrt{\frac{1}{\sigma^2}\cdot\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{n-1}}} = \frac{Z}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{X_i - \overline{X}}{\sigma}\right)^2}}$$

Prop. 3.

$X \sim \mathrm{N}(0,1)$, $Y \sim \mathrm{Ga}\left(\frac{1}{2}, \frac{n}{2}\right)$, independently.

Then, $\frac{X}{\sqrt{\frac{Y}{n}}}$ follows.

t-distribution
with n degrees
of freedom

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\ \Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}} \qquad (-\infty < x < \infty).$$

t-distribution and χ^2distribution [William Gosset]

$$t = Z\frac{\sigma}{s} = \frac{Z}{\sqrt{\frac{s^2}{\sigma^2}}} = \frac{Z}{\sqrt{\frac{1}{\sigma^2}\cdot\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{n-1}}} = \frac{Z}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{X_i - \overline{X}}{\sigma}\right)^2}}$$

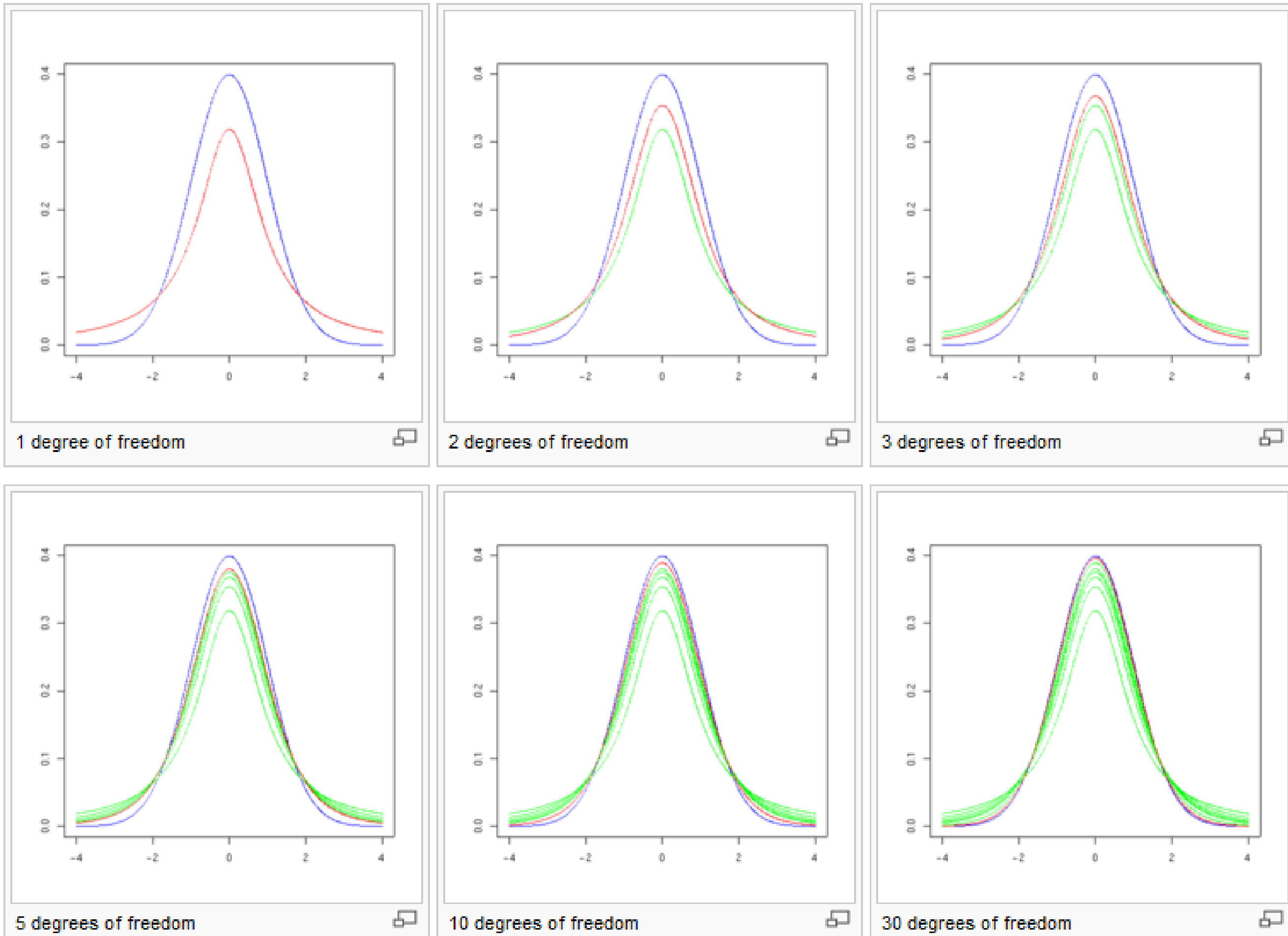
Thm.

t follows the t-distribution with $n-1$ degrees, i.e.,

$$f_t(x) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{(n-1)\pi}\ \Gamma\left(\frac{n-1}{2}\right)}\left(1+\frac{x^2}{n-1}\right)^{-\frac{n}{2}} \qquad (-\infty < x < \infty)$$

Student’s t distribution

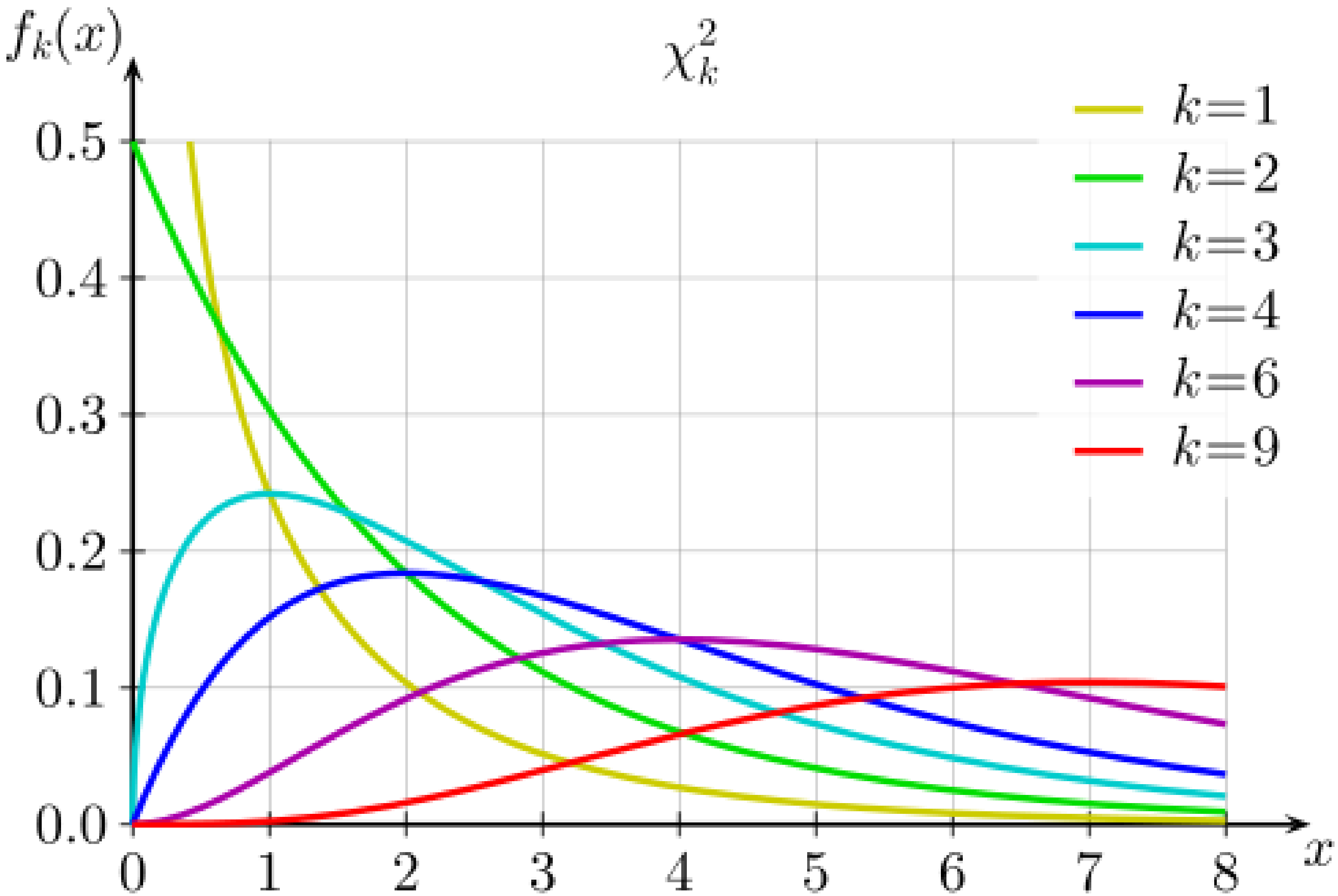
Density of the *t*-distribution (red) for 1, 2, 3, 5, 10, and 30 df compared to the standard normal distribution (blue). Previous plots shown in green.



Wikipedia: Student’s t distribution

http://en.wikipedia.org/wiki/Student%27s_t-distribution

χ^2 分布



Wikipedia: Chi-squared distribution

http://en.wikipedia.org/wiki/Chi-squared_distribution

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- interval estimation (区間推定)
- hypothesis testing (仮説検定)
  - **t-test**
  - **χ^2-test**

t-test (t検定)

estimation of μ (expect.)

Statistical inference

Example 1

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You bought 6 eggs in a shop.

	1	2	3	4	5	6
weight[g]	64.3	70.4	63.2	67.8	71.3	60.8

How large are eggs sold in this shop?

$\bar{X}$ = 66.3[g], s^2 = 17.584[g^2]

Is the clerk honest?

t-test (t検定)

t-test

Given samples $X_1 = a_1, \ldots, X_n = a_n$.

Q: Does a value b estimate $\mathrm{E}[X]$?

Since $t := \frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}}$ follows t distribution with degree n-1,

$$\Pr[\text{null hypo.}: \mathrm{E}[X] = b] = \Pr\left[\left|\bar{X} - b\right| \geq |\bar{a} - b| \mid E[X] = b\right]$$

$$= \int_{-\infty}^{-\frac{|\bar{a}-b|}{\sqrt{s^2/n}}} f_t(x)dx + \int_{\frac{|\bar{a}-b|}{\sqrt{s^2/n}}}^{\infty} f_t(x)\mathrm{d}x \quad (1$$

Claim

If $(1) < \alpha \Rightarrow$ it rejects $\mathrm{E}[X] = b$

If $(1) \geq \alpha \Rightarrow$ it fails to reject $\mathrm{E}[X] = b$

Student's t-statistics (スチューデントのt統計量)

Example 1

A clerk says "our eggs are big. 70[g] in average."

$\bar{X}$ = 66.3[g], s^2 = 17.584[g^2] for 6 eggs.

Let assume μ = 70.0 ~~Suppose σ^2=18.0 for simplicity.~~

Let $t := \frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}}$, ← $Z_n := \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$ in Cent. limit. Thm.

where $s^2 := \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1}$ (unbiased estimator of σ^2).

Then t, follows t distribution with degree $n-1$

$$f_t(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\ \Gamma\left(\frac{n}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}} \qquad (-\infty < x < \infty).$$

Statistical inference

Example 1

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Let assume μ = 70.0 ~~Suppose σ^2=18.0 for simplicity.~~

Let t^* (>0) satisfy

$$\Pr\left[-t^* \leq \frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}} \leq t^*\right] = \int_{-t^*}^{t^*} f_t(x)\mathrm{d}x \geq 0.95$$

… and we see that $t^* = 2.571$ (see t-distribution table).

Statistical inference

Example 1

A clerk says “our eggs are big. 70[g] in average.”

$\bar{X}$ = 66.3[g], s^2 = 17.584[g^2] for 6 eggs.

Let assume μ = 70.0 ~~Suppose σ^2=18.0 for simplicity.~~

$\bar{X}$ = 66.3[g]

σ^2=17.584

n = 6

z^*=2.571

$$\left|\frac{\bar{X}-\mu}{\sqrt{\frac{s^2}{n}}}\right| = \left|\frac{66.3-70}{\sqrt{\frac{17.584}{6}}}\right| = 2.161 < t^* = 2.571$$

It fails to reject the null hypothesis μ = 70.0 with significant level 5%

(帰無仮説 μ = 70.0 は有意水準5%で棄却されない。)

Todays topics

- interval estimation (区間推定)
- hypothesis testing (仮説検定)
  - **t-test**
  - **χ^2-test**

χ^2-test (χ^2検定)

estimation of σ^2 (variance.)

χ^2-test (χ^2検定)

χ^2-test

Given samples $X_1 = a_1, \ldots X_n = a_n$.

Q: Does a value c^2 estimate $\mathrm{Var}[X]$?

Since $S := \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2}$ follows

χ^2 distribution with n-1 degrees of freedom,

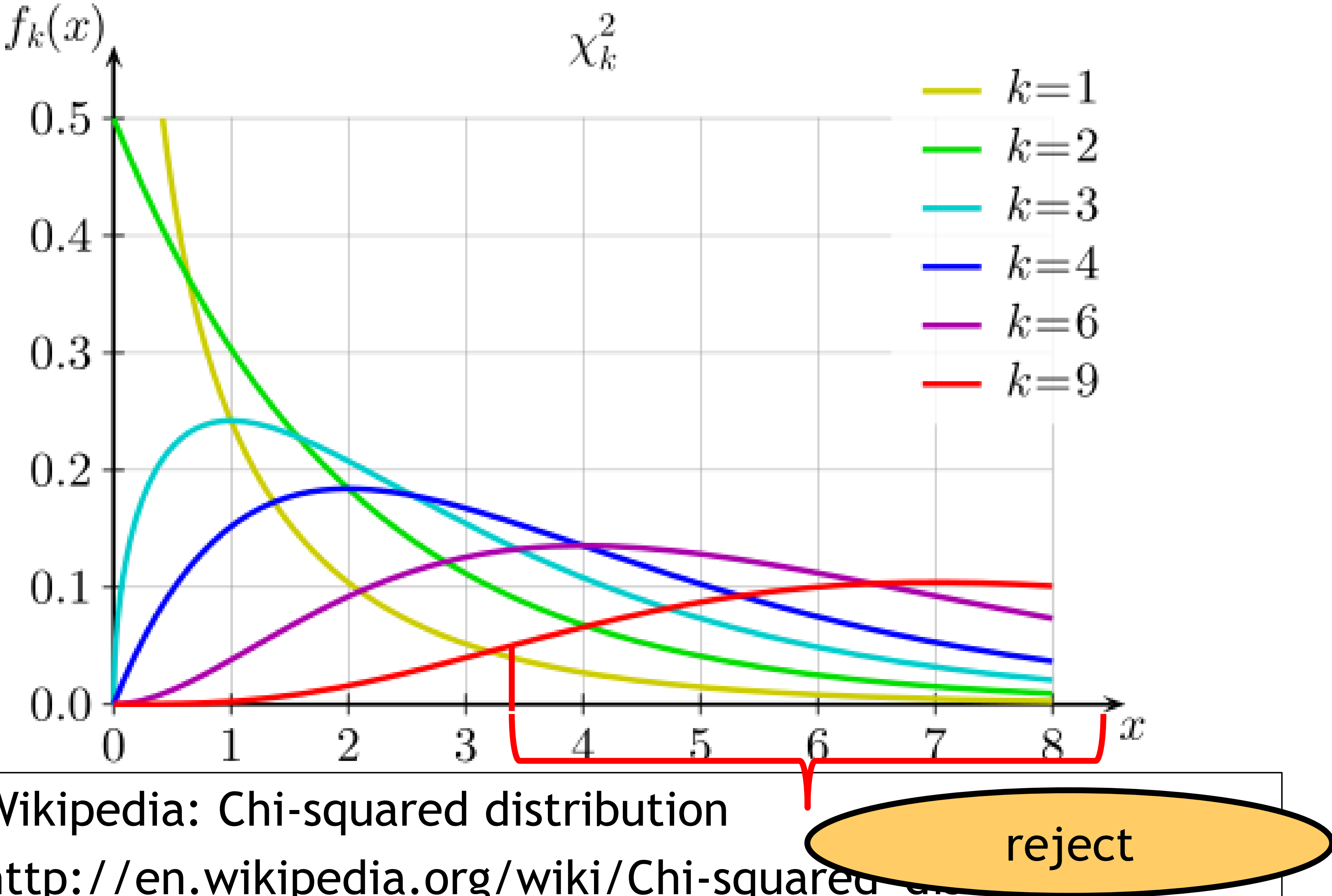
$$\mathrm{Pr}[\text{null hypothesis: } \mathrm{Var}[X] = c^2] = \mathrm{Pr}[S \geq c^2 \mid \mathrm{Var}[X] = c^2]$$
$$= \int_{c^2}^{\infty} f_{\chi^2}(x)\mathrm{d}x \qquad (2)$$

Claim

If $(2) < \alpha \Rightarrow$ it rejects $\mathrm{Var}[X] = \mathrm{c}^2$

If $(2) \geq \alpha \Rightarrow$ it fails to reject $\mathrm{Var}[X] = \mathrm{c}^2$

χ^2 分布



Wikipedia: Chi-squared distribution

http://en.wikipedia.org/wiki/Chi-squared_dis

χ^2-test (χ^2検定) Example

χ^2-test

Suppose the sample variance of weights of 10 balls is 0.35.

Is this smaller than the prescribed value 0.2?

Discuss with significant level 5%

null hypothesis (帰無仮説)

Var[X] ≤ 0.2

right 5%

$$S := \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2} = \frac{(n-1)s^2}{\sigma^2} = \frac{(10-1)\times 0.35}{0.2} = 15.75 < 16.919$$

Claim

It fails to reject the null hypothesis with significant level 5%.

(有意水準5%で帰無仮説は棄却されない)

χ^2-test (χ^2検定) Example

χ^2-test

Suppose the sample variance of weights of 100 balls is 0.26.

Is this smaller than the prescribed value 0.2?

Discuss with significant level 5%

null hypothesis (帰無仮説)

Var[X] $\leq$ 0.2

right 5%

$$S := \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2} = \frac{(n-1)s^2}{\sigma^2} = \frac{(100-1) \times 0.26}{0.2} = 128.7 > 124.34$$

Claim

It rejects the null hypothesis with significant level 5%.

(有意水準5%で帰無仮説は棄却される)

z-test: Normal Distribution

t-test: t distribution such as expectation

χ^2-test: χ^2 distribution such as variance

F-test: F distribution such as ratio of variance