Lesson 5

Law of Large Numbers

Todays topics

- law of large numbers
- central limit theorem
- generating functions

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Midterm exam (中間試験)

Date/time: November 16 (11/16), 8:40-

Place (場所): here (工学部第一講義室).

Topics (範囲):

Related to the classes during Oct. 10 – Nov. 9,

about fundamental probability.

(確率論基礎: 11/9の講義内容まで)

check the course page (講義ページを参照のこと)

http://tcslab.csce.kyushu-u.ac.jp/~kijima/

books, notes, resumes, computers, etc.

are prohibited to use at exam. (持ち込み不可)

Law of Large number

Law of large numbers (大数の法則)

Def.

A series $\{Y_n\}$ converges Y in probability (Yに確率収束する), if

$\forall \varepsilon > 0, \lim_{n\to\infty} \Pr[|Y_n - Y| < \varepsilon] = 1$

independent and identically distributed (独立同一分布)

Thm. (low of large numbers; 大数の法則)

Suppose $X_1, \ldots, X_n$ are i.i.d., w/ expectation μ, and variance σ^2,

then $\frac{X_1+\cdots+X_n}{n}$ converge μ in probability;

i.e., $\forall \varepsilon > 0, \lim_{n\to\infty} \Pr\left[\left|\frac{X_1+\cdots+X_n}{n} - \mu\right| < \varepsilon\right] = 1$

Thm. (low of large numbers; 大数の法則)

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then $\frac{X_1+\cdots+X_n}{n}$ converge μ in probability;

i.e., $\forall\varepsilon > 0,\ \lim_{n\to\infty} \Pr\left[\left|\frac{X_1+\cdots+X_n}{n} - \mu\right| < \varepsilon\right] = 1$

Proof.

Let $Y_n \coloneqq \frac{X_1+\cdots+X_n}{n}$, for simplicity.

✓ $\mathrm{E}[Y_n] = \mathrm{E}\left[\frac{X_1+\cdots+X_n}{n}\right] = \frac{\mathrm{E}[X_1]+\cdots+\mathrm{E}[X_n]}{n} = \frac{\mu+\cdots+\mu}{n} = \mu$

✓ $\mathrm{Var}[Y_n] = \mathrm{Var}\left[\frac{X_1+\cdots+X_n}{n}\right] = \frac{\mathrm{Var}[X_1]+\cdots+\mathrm{Var}[X_n]}{n^2} = \frac{\sigma^2+\cdots+\sigma^2}{n^2} = \frac{\sigma^2}{n}$

By Chebyshev's inequality,

$$\forall\varepsilon > 0,\ \forall n > 0,\ \Pr\left[\left|\frac{X_1+\cdots+X_n}{n} - \mu\right| \geq \varepsilon\right] \leq \frac{\mathrm{Var}[Y_n]}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}$$

$$\forall\varepsilon > 0,\ \Pr\left[\left|\frac{X_1+\cdots+X_n}{n} - \mu\right| < \varepsilon\right] \geq 1 - \frac{\sigma^2}{n\varepsilon^2} \xrightarrow{n\to\infty} 1$$

Central Limit Theorem

Central Limit Theorem (中心極限定理)

Def.

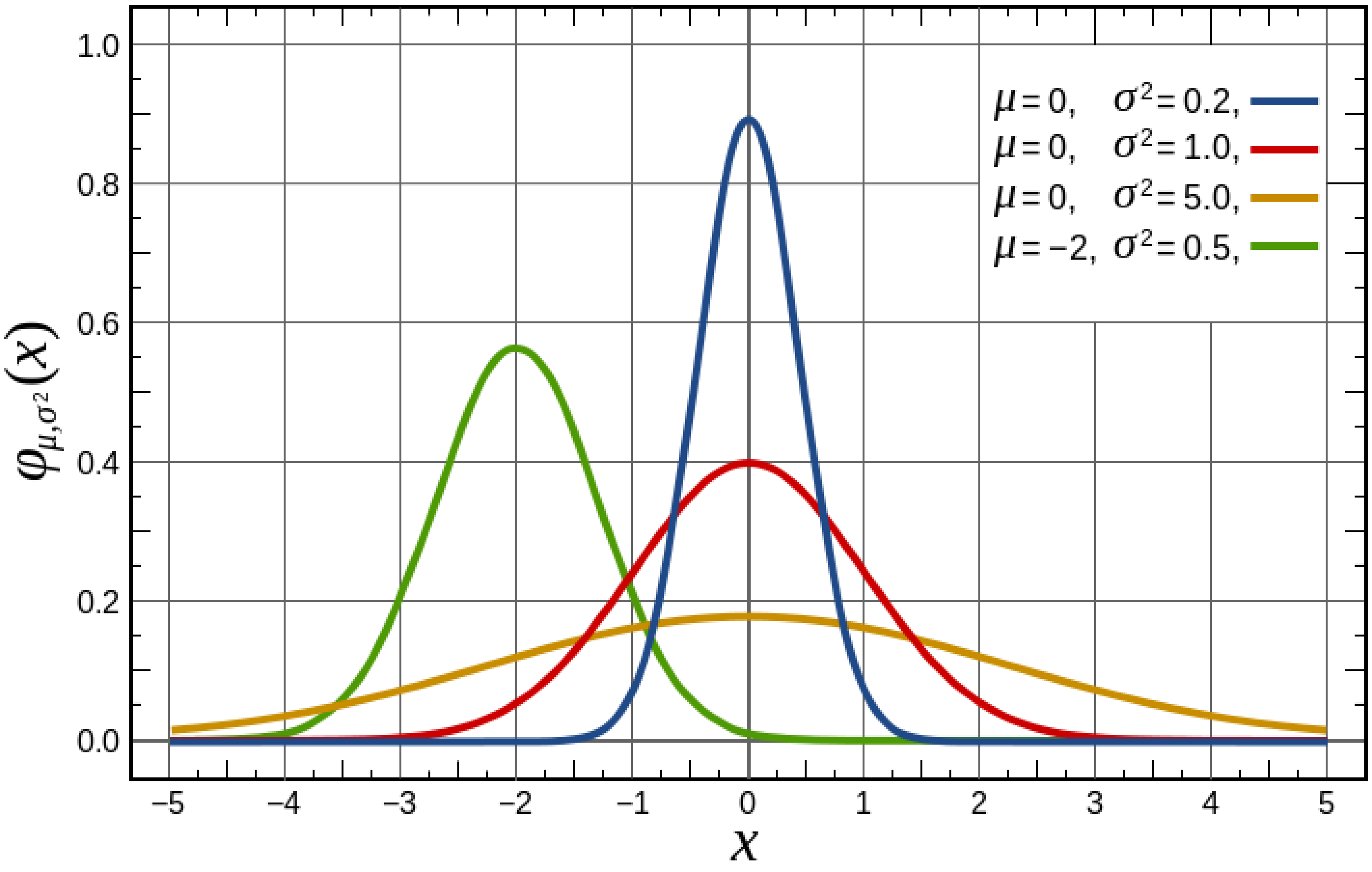
A series $\{Y_n\}$ w/ distribution functions $\{F_n\}$ converges Y in distribution (Yに分布収束する), if $\lim_{n\to\infty} F_n = F$ where F is the distr. func. of Y.

Thm. Central limit theorem

Suppose $X_1, \dots, X_n$ are i.i.d., w/ expectation μ, and variance σ^2, then $Z_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$ converges to $\mathrm{N}(0,1)$ in distribution.

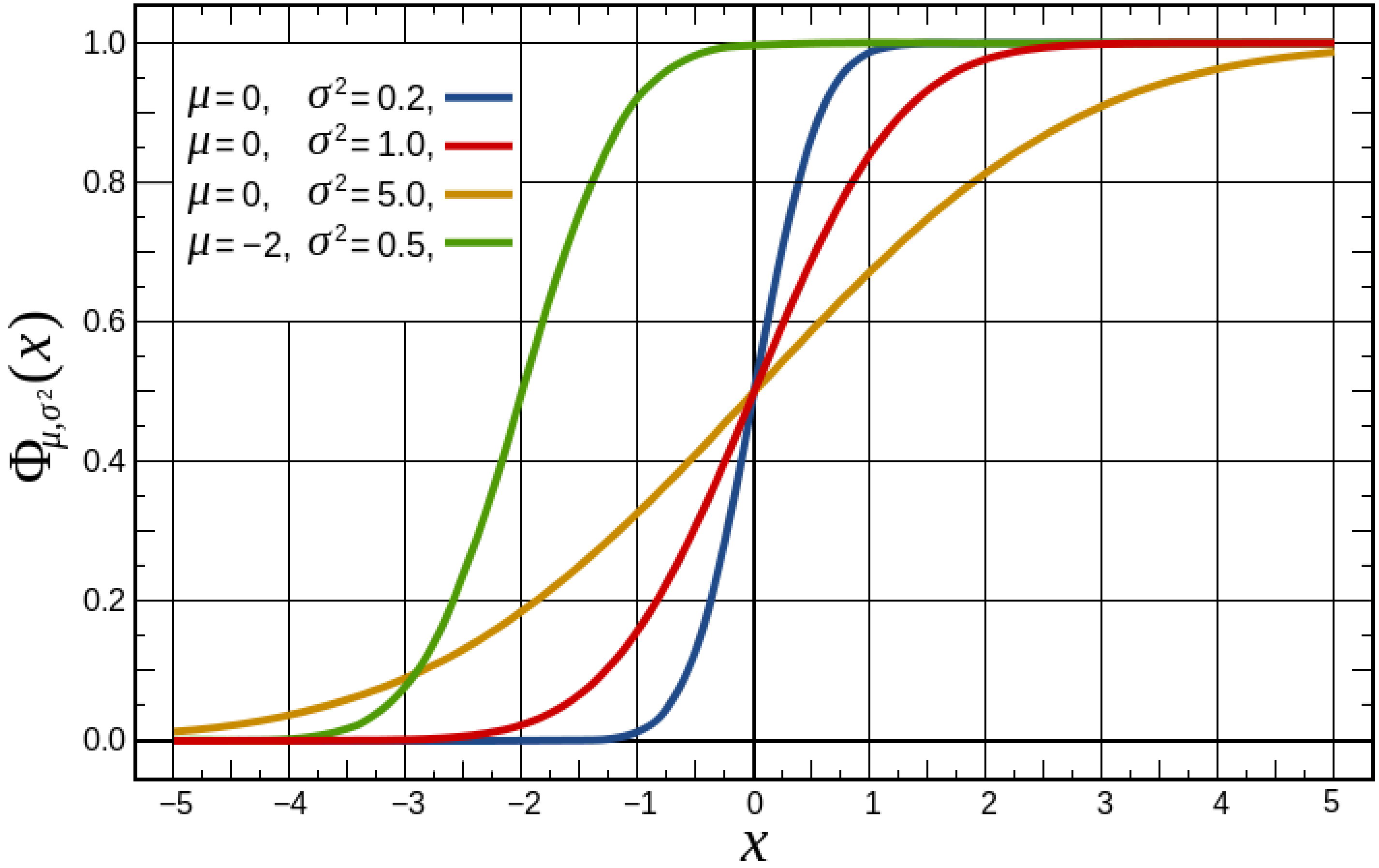
i.e., $\lim_{n\to\infty} \Pr[Z_n < z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x$

pdf of normal distribution



http://en.wikipedia.org/wiki/Normal_distribution

Distr. func. of normal distrbution



http://en.wikipedia.org/wiki/Normal_distribution

Central Limit Theorem (中心極限定理)

Suppose $X_1, \dots, X_n$ are i.i.d., w/ expectation μ, and variance σ^2,

then $Z_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$ converges to $\mathrm{N}(0,1)$ in distribution.

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Before the proof…

Central Limit Theorem (中心極限定理)

Suppose $X_1, \ldots, X_n$ are i.i.d., w/ expectation μ, and variance σ^2,

then $Z_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$ converges to $\mathrm{N}(0,1)$ in distribution.

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Corollary

Suppose $X_1, \ldots, X_n$ are i.i.d., w/ expectation μ, and variance σ^2,

then $\frac{X_1 + \cdots + X_n}{n}$ converges to $\mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ in distribution.

Affine transform. of a normal distribution

Prop.

Let $a \in \mathbf{R}_{>0}$, $b \in \mathbf{R}$. Suppose that $X \sim N(\mu, \sigma^2)$, and let $Y := aX + b$. Then, $Y \sim N(a\mu + b, a^2\sigma^2)$.

Affine transform. of a random variable

Affine transform. of a discrete random variable

Prop.

Let $a \in \mathbf{R}_{>0}$, $b \in \mathbf{R}$. Suppose that X is a discrete random variable w/ pmf. $f_X(x)$, and let $Y := aX + b$. Then, Y follows the pmf.

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)$$

Proof.

Since $Y := aX + b$,

$$[Y = y] \Leftrightarrow [aX + b = y] \Leftrightarrow \left[X = \frac{y-b}{a}\right]$$

i.e.,

$\Pr[Y = y]$ $\qquad$ $\Pr\left[X = \frac{y-b}{a}\right]$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right).$$

Affine transform. of a continuous random variable

Prop.

Let $a \in \mathbf{R}_{>0}$, $b \in \mathbf{R}$. Suppose that X is a continuous random variable w/ pdf $f_X(x)$, and let $Y := aX + b$. Then, Y follows the pdf.

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right).$$

Proof.

Since $Y := aX + b$,

$$[Y \leq y] \Leftrightarrow [aX + b \leq y] \Leftrightarrow \left[X \leq \frac{y-b}{a}\right]$$

$$\Pr[Y \leq y] = \Pr\left[X \leq \frac{y-b}{a}\right]$$

i.e.,

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right).$$

By differentiating the both sides, we obtain

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right).$$

Affine transform. of a normal distribution

Prop.

Let $a \in \mathbf{R}_{>0}$, $b \in \mathbf{R}$. Suppose that $X \sim N(\mu, \sigma^2)$, and let $Y := aX + b$. Then, $Y \sim N(a\mu + b, a^2\sigma^2)$.

Affine transform. of a normal distribution

Prop.
Let $a \in \mathbf{R}_{>0}$, $b \in \mathbf{R}$. Suppose that $X \sim N(\mu, \sigma^2)$, and let $Y := aX + b$. Then, $Y \sim N(a\mu + b, a^2\sigma^2)$.

Another proof. Since $\Pr[Y \le y] = \Pr\left[X \le \frac{y-b}{a}\right]$,

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right) = \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) \mathrm{d}t \qquad (*)$$

let $s = at + b$, then $\mathrm{d}s = a\mathrm{d}t$ and

t	$-\infty \to \frac{y-b}{a}$
$s = at + b$	$-\infty \to y$

$$(*) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\left(\frac{t-b}{a}-\mu\right)^2}{2\sigma^2}\right) \frac{1}{a} \mathrm{d}s$$

$$= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\left(\frac{s-b}{a}-\mu\right)^2}{2\sigma^2}\right) \frac{1}{a} \mathrm{d}s$$

$$= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}a\sigma} \exp\left(-\frac{(s-(a\mu+b))^2}{2a^2\sigma^2}\right) \mathrm{d}s$$

⇨ density function of $N(a\mu + b, (a\sigma)^2)$

Central Limit Theorem (中心極限定理)

Suppose $X_1, \dots, X_n$ are i.i.d., w/ expectation μ, and variance σ^2,

then $Z_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$ converges to $\mathrm{N}(0,1)$ in distribution.

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Corollary

Suppose $X_1, \dots, X_n$ are i.i.d., w/ expectation μ, and variance σ^2,

then $\frac{X_1 + \cdots + X_n}{n}$ converges to $\mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ in distribution.

Sum of random variables

Sum of random variables

Observation

Let X, Y be independent integer random variables and let f_X, f_Y be their probability functions, resp.
Let $Z := X + Y$, then the probability function f_Z of Z is

$$f_Z(t) = \sum_{s=-\infty}^{\infty} f_X(s) f_Y(t-s)$$

Observation

Let X, Y be independent continuous random variables and let f_X, f_Y be their density functions, resp.
Let $Z := X + Y$, then the density function f_Z of Z is

$$f_Z(t) = \int_{-\infty}^{\infty} f_X(s) f_Y(t-s) \mathrm{d}s$$

Ex. Gamma distr.

Proposition 2

Suppose $X_1 \sim \mathrm{G}(\alpha, \nu_1)$ and $X_2 \sim \mathrm{G}(\alpha, \nu_2)$ are independent,

then $Y := X_1 + X_2 \sim \mathrm{G}(\alpha, \nu_1 + \nu_2)$

Assume ν_1 and ν_2 are Integers, for convenience.

omit the proof here.

Ex. Gamma distr.

Proposition 2

Suppose $X_1 \sim \mathrm{G}(\alpha, \nu_1)$ and $X_2 \sim \mathrm{G}(\alpha, \nu_2)$ are independent,

then $Y := X_1 + X_2 \sim \mathrm{G}(\alpha, \nu_1 + \nu_2)$

Assume ν_1 and ν_2 are Integers, for convenience.

$$
\begin{aligned}
f_X(x) &= \int_0^x f_{X_1}(t) f_{X_2}(x-t)\,\mathrm{d}t \\
&= \int_0^x \frac{1}{\Gamma(\nu_1)} \alpha^{\nu_1} t^{\nu_1 - 1} \mathrm{e}^{-\alpha t} \frac{1}{\Gamma(\nu_2)} \alpha^{\nu_2} (x-t)^{\nu_2 - 1} \mathrm{e}^{-\alpha(x-t)}\,\mathrm{d}t \\
&= \int_0^x \frac{\Gamma(\nu_1 + \nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \cdot \frac{1}{\Gamma(\nu_1 + \nu_2)} \alpha^{\nu_1 + \nu_2} \mathrm{e}^{-\alpha x} t^{\nu_1 - 1} (x-t)^{\nu_2 - 1}\,\mathrm{d}t \\
&= \frac{1}{\Gamma(\nu_1 + \nu_2)} \alpha^{\nu_1 + \nu_2} \mathrm{e}^{-\alpha x} \int_0^x \frac{\Gamma(\nu_1 + \nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} t^{\nu_1 - 1} (x-t)^{\nu_2 - 1}\,\mathrm{d}t
\end{aligned}
$$

Ex. Gamma distr.

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then $Y := X_1 + X_2 \sim \mathrm{G}(\alpha, \nu_1 + \nu_2)$

Assume ν_1 and ν_2 are Integers, for convenience.

let $s = t/x$, i.e., $t = xs$, then $\mathrm{d}t = x\mathrm{d}s$ and

$$\int_0^x \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} t^{\nu_1-1}(x-t)^{\nu_2-1}\mathrm{d}t$$

$$= \int_0^1 \frac{(\nu_1+\nu_2)!}{\nu_1!\,\nu_2!}(xs)^{\nu_1-1}(x-xs)^{\nu_2-1}x\mathrm{d}s$$

$$= x^{\nu_1+\nu_2-1}\int_0^1 \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} s^{\nu_1-1}(1-s)^{\nu_2-1}\mathrm{d}s$$

$$= x^{\nu_1+\nu_2-1}$$

Ex. Normal distr.

Suppose $X \sim \mathrm{N}(\mu_1, \sigma_1^2)$, $Y \sim \mathrm{N}(\mu_2, \sigma_2^2)$ are independent.
Compute the density function of $Z := X + Y$.

$$
\begin{aligned}
f_Z(x) &= \int_{-\infty}^{\infty} f_X(t) f_Y(x-t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left(-\frac{(t-\mu_1)^2}{\sigma_1^2}\right) \frac{1}{\sqrt{2\pi}\,\sigma_2} \exp\left(-\frac{(x-t-\mu_2)^2}{\sigma_2^2}\right) \mathrm{d} \\
&= \cdots
\end{aligned}
$$

Hard!

Generating Functions

a technique for sum of random variables

a technique for convolution

Generating functions

<u>Def. (probability generating function; 確率母関数)</u>

Let X be a nonnegative integer valued random variable, the *probability generating function* is defined by

$g(z) := E[z^X] = \sum_{k=0}^{+\infty} z^k f_X(k)$ $(-1 < z < 1)$

<u>Observations.</u>

1. $\sum_{k=0}^{+\infty} |z^k f_X(k)| < \sum_{k=0}^{+\infty} |z^k| = \frac{1}{1-z} < +\infty$
2. $g'(z) = \sum_{k=0}^{+\infty} \frac{d}{dz} z^k f_X(k) = \sum_{k=0}^{+\infty} k z^{k-1} f_X(k)$
3. $g''(z) = \frac{d}{dz} g'(z) = \sum_{k=0}^{+\infty} \frac{d}{dz} k z^{k-1} f_X(k) = \sum_{k=0}^{+\infty} k(k-1) z^{k-2} f_X(k)$
4. $g(1) = 1$.
5. $g'(1) = E[X]$.
6. $g''(1) = E[X(X-1)] = E[X^2] - E[X]$
7. $Var[X] = g''(1) + g'(1) - (g'(1))^2$

Generating functions

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Thm.

Let X, Y be independent nonnegative int. r.v., and let $g_X(z)$, $g_Y(z)$ be their prob. gen. func.

Let Z := X+Y, then its prob. gen. func. $g_Z(z)$ satisfies

$g_Z(z) = g_X(z)\, g_Y(z)$

Thm.

Let X, Y be independent nonnegative int. r.v., and let $g_X(z)$, $g_Y(z)$ be their prob. gen. func.

Let Z := X+Y, then its prob. gen. func. $g_Z(z)$ satisfies

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proof

$\Pr[X+Y=k] = \sum_{l=0}^{k} \Pr[X = l, Y = k - l]$

$= \sum_{l=0}^{k} \Pr[X = l] \Pr[y = k - l]$

$g_Z(z) = \sum_{k=0}^{+\infty} \Pr[X + Y = k]\, z^k$

$= \sum_{k=0}^{+\infty} \sum_{l=0}^{k} \Pr[X = l] \Pr[y = k - l] z^l z^{k-l}$

Thm.

Let X, Y be independent nonnegative int. r.v., and let $g_X(z)$, $g_Y(z)$ be their prob. gen. func.

Let Z := X+Y, then its prob. gen. func. $g_Z(z)$ satisfies

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(l,k-l)	l=0	1	2	3	4
k=0	(0,0)				
1	(0,1)	(1,0)			
2	(0,2)	(1,1)	(2,0)		
3	(0,3)	(1,2)	(2,1)	(3,0)	
4	(0,4)	(1,3)	(2,2)	(3,1)	(4,0)

proof

$\Pr[X+Y=k] = \sum_{l=0}^{k} \Pr[X = l, Y =$

$= \sum_{l=0}^{k} \Pr[X = l]\, \Pr$

$g_Z(z) = \sum_{k=0}^{+\infty} \Pr[X + Y = k]\, z^k$

$= \sum_{k=0}^{+\infty} \sum_{l=0}^{k} \Pr[X = l]\, \Pr[y = k - l] z^l z^{k-l}$

$= \sum_{l=0}^{+\infty} \Pr[X = l]\, z^l \sum_{k=l}^{+\infty} \Pr[y = k - l] z^{k-l}$

$= \sum_{l=0}^{+\infty} \Pr[X = l]\, z^l \sum_{l'=0}^{+\infty} \Pr[y = l']\, z^{l'}$

$= (\sum_{l=0}^{+\infty} \Pr[X = l]\, z^l)\, g_Y(z)$

$= g_X(z)\, g_Y(z)$

$\blacksquare$

Generating functions

Def. (probability generating function; 確率母関数)

Let X be a nonnegative integer valued random variable, the *probability generating function* is defined by

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Let X be a nonnegative integer valued random variable,

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$g(z) := E[z^X] = \sum_{k=0}^{+\infty} z^k f_X(k)$ $(-1 < z < 1)$

Def. (moment generating function; 積率母関数)

Let X be a random variable,

the *moment generating function* is defined by

$M(\theta) := E[e^{\theta X}]$ $(\theta \in R)$

Def. (Characteristic function; 特性ベクトル)

Let X be a random variable,

the *characteristic function* is defined by

$\varphi(t) := E[e^{itX}]$ $(t \in R)$

Generating functions

Def. (moment generating function; 積率母関数)

Let X be a random variable,

the probability generating function is defined by

$M(\theta) := E[e^{\theta X}] \quad (\theta \in R)$

Observations.

1. $M^{(k)}(0) = E[X^k]$

Generating functions

<u>Thm.</u>

Let X, Y be independent nonnegative int. r.v., and

let $g_X(z)$, $g_Y(z)$ be their prob. gen. func.

Let Z := X+Y, then its prob. gen. func. $g_Z(z)$ satisfies

$$g_Z(z) = g_X(z)\, g_Y(z)$$

<u>Thm.</u>

Let X, Y be independent r.v.,

let $M_X(\theta)$, $M_Y(\theta)$ be their moment gen. func.,

(let $\varphi_X(t)$, $\varphi_Y(t)$ be their characteristic func.)

Let Z := X+Y, then

its mom. gen. func. $M_Z(\theta)$ (char. func. $\varphi_Z(t)$) satisfies

$$M_Z(\theta) = M_X(\theta)\, M_Y(\theta) \qquad (\varphi_Z(t) = \varphi_X(t)\, \varphi_Y(t))$$

Ex. compute the moment gen. func. of $X \sim N(\mu,\sigma^2)$

Ex. compute the moment gen. func. of X ~N(μ,σ²)

$$\mathrm{E}[\mathrm{e}^{tX}] = \int_{-\infty}^{\infty} \mathrm{e}^{tx} \frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-\frac{x^2-2(\mu+t\sigma^2)x+\mu^2}{2\sigma^2}} \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-\frac{\left(x-(\mu+t\sigma^2)\right)^2}{2\sigma^2}} \mathrm{e}^{-\frac{(\mu+t\sigma^2)-\mu^2}{2\sigma^2}} \mathrm{d}x$$

$$= \mathrm{e}^{-\frac{(\mu+t\sigma^2)-\mu^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-\frac{\left(x-(\mu+t\sigma^2)\right)^2}{2\sigma^2}} \mathrm{d}x$$

$$= \mathrm{e}^{-\frac{(\mu+t\sigma^2)-\mu^2}{2\sigma^2}}$$

$$= \mathrm{e}^{-\mu t+\frac{\sigma^2}{2}t^2}$$

Ex.

Suppose X ~N(μ_1,σ_1^2), Y ~N(μ_2,σ_2^2) are independent.
Compute the density function of Z:=X+Y.

$$M_z(\theta) = M_X(\theta)M_Y(\theta)$$
$$= \mathrm{e}^{-\mu_1 t+\frac{\sigma_1^2}{2}t^2}\mathrm{e}^{-\mu_2 t+\frac{\sigma_2^2}{2}t^2}$$
$$= \mathrm{e}^{-(\mu_1+\mu_2)t+\frac{\sigma_1^2+\sigma_2^2}{2}t^2}$$

Hence Z ~N(μ_1+μ_2,σ_1^2 +σ_2^2),

$$f_Z(x) = \frac{1}{\sqrt{2\pi}(\sigma_1^2+\sigma_2^2)}\exp\left(-\frac{(x-(\mu_1+\mu_2)^2)}{2(\sigma_1^2+\sigma_2^2)}\right)$$

Central Limit Theorem

Central Limit Theorem (中心極限定理)

Def.

A series $\{Y_n\}$ w/ distribution functions $\{F_n\}$

converges Y in distribution (Yに分布収束する), if

$\lim_{n\to\infty} F_n = F$ where F is the distr. func. of Y.

Thm. Central limit theorem

Suppose $X_1, \dots, X_n$ are i.i.d., w/ expectation μ, and variance σ^2,

then $Z_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$ converges to $\mathrm{N}(0,1)$ in distribution.

i.e., $\lim_{n\to\infty} \Pr[Z_n < z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x$

Central Limit Theorem (中心極限定理)

Suppose $X_1, \ldots, X_n$ are i.i.d., w/ expectation μ, and variance σ^2,

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i.e., $\lim_{n \to \infty} \Pr[Z_n < z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x$

Let $Y_i := \frac{X_i - \mu}{\sigma}$, then $Z_n = \frac{Y_1 + \cdots + Y_n}{\sqrt{n}}$.

Let $\varphi(t) = \mathrm{E}[\mathrm{e}^{tY}]$ (i.e., moment gen. func. of Y_i), then

$$\varphi(0) = 1$$

$$\varphi'(0) = E[Y] = 0$$

$$\varphi''(0) = E[Y^2] = E\left[\frac{(X_i - \mu)^2}{\sigma^2}\right] = 1$$

Central Limit Theorem (中心極限定理)

Suppose $X_1, \dots, X_n$ are i.i.d., w/ expectation μ, and variance σ^2,

then $Z_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$ converges to $\mathrm{N}(0,1)$ in distribution.

i.e., $\lim_{n \to \infty} \Pr[Z_n < z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x$

$$
\begin{aligned}
\varphi_{Z_n}(t) &= \mathrm{E}\left[\exp\left(\frac{t(Y_1 + \cdots + Y_n)}{\sqrt{n}}\right)\right] \\
&= \mathrm{E}\left[\exp\left(\frac{tY_1}{\sqrt{n}}\right)\right] \cdots \mathrm{E}\left[\exp\left(\frac{tY_n}{\sqrt{n}}\right)\right] \\
&= \left(\varphi\left(\frac{t}{\sqrt{n}}\right)\right)^n \\
&= \left(\varphi(0) + \varphi'(0)\frac{t}{\sqrt{n}} + \frac{1}{2}\varphi''(0)\left(\frac{t}{\sqrt{n}}\right)^2 + \cdots\right)^n \\
&= \left(1 + \frac{1}{2}\left(\frac{t}{\sqrt{n}}\right)^2 + o\left(\frac{1}{n}\right)\right)^n \xrightarrow{n \to \infty} \mathrm{e}^{\frac{t^2}{2}}
\end{aligned}
$$

since independent

Central Limit Theorem (中心極限定理)

Suppose $X_1, \dots, X_n$ are i.i.d., w/ expectation μ, and variance σ^2,

then $Z_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$ converges to $\mathrm{N}(0,1)$ in distribution.

i.e., $\lim_{n\to\infty} \Pr[Z_n < z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x$

Corollary

Suppose $X_1, \dots, X_n$ are i.i.d., w/ expectation μ, and variance σ^2,

then $\frac{X_1 + \cdots + X_n}{n}$ converges to $\mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ in distribution.

Exercises

Normal distr. (正規分布) N(μ, σ²)

$\Omega = (-\infty, +\infty)$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

for x $(-\infty < x < \infty)$.

Question 1.

Expectation of $X \sim \mathrm{N}(\mu, \sigma^2)$ is ??

Variance of $X \sim N(\mu, \sigma^2)$ is ??

Ex. expectation of normal distr.

Proposition 1.

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\mathrm{E}[X] = \mu$, $\mathrm{Var}[X] = \sigma^2$

proof

$$z = \frac{x-\mu}{\sigma}$$

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} (z\sigma + \mu) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2\right) \mathrm{d}z \\
&= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \exp\left(-\frac{1}{2} z^2\right) \mathrm{d}z + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2\right) \mathrm{d}z \\
&= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \exp\left(-\frac{1}{2} z^2\right) \mathrm{d}z + \mu \\
&= \mu
\end{aligned}
$$

Ex. expectation of normal distr.

Proposition 1.

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\mathrm{E}[X] = \mu$, $\mathrm{Var}[X] = \sigma^2$

proof

$$z = \frac{x-\mu}{\sigma}$$

$$\begin{aligned}
\mathrm{E}[(X-\mu)^2] &= \int_{-\infty}^{+\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \mathrm{d}x \\
&= \int_{-\infty}^{+\infty} z^2 \frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2\right) \sigma \,\mathrm{d}z \\
&= \sigma^2 \int_{-\infty}^{+\infty} (-z)\,(-z) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2\right) \mathrm{d}x \\
&= \sigma^2 \left(\left[(-z) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2\right) \right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2\right) \mathrm{d}x \right) \\
&= \sigma^2 \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2\right) \mathrm{d}x \\
&= \sigma^2
\end{aligned}$$

Square of standard normal random variable

Question 2

If $X \sim \mathrm{N}(0,1)$ then $X^2 \sim$??

Square of standard normal random variable

Proposition 2

If $X \sim \mathrm{N}(0,1)$ then $X^2 \sim \mathrm{G}\left(\frac{1}{2},\frac{1}{2}\right)$

Gamma distr. $G(\alpha,\nu)$

$$f_G(x) = \frac{1}{\Gamma(k)}\alpha^{\nu}e^{-\alpha x}x^{1-\nu}$$

proof.

Gamma distr. (ガンマ分布) G(α,ν) (α>0, ν>0)

Ω=[0, +∞)

$$f(x) = \frac{1}{\Gamma(\nu)} \alpha^{\nu} x^{\nu-1} \mathrm{e}^{-\alpha x} \quad (x \geq 0)$$

where

$$\Gamma(\nu) = \int_0^{\infty} t^{\nu-1} \mathrm{e}^{-t} \mathrm{d}t$$

Remark that

$$\Gamma(1) = 1$$

$$\Gamma(\nu) = (\nu - 1) \cdot \Gamma(\nu - 1)$$

In case of $\nu = 1,2, \ldots,$

$$\Gamma(\nu) = \nu!$$

Expectation of X ~ G(α,ν) is ?

Variance of X ~ G(α,ν) is ?

Beta distr. (ベータ分布) Be(α,β) ($\alpha > 0, \beta > 0$)

Ω=[0, 1]

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

where

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \mathrm{d}t$$

Expectation of X ~ Be(α,β) is ?

Variance of X ~ Be(α,β) is ?

remark that

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Square of standard normal random variable

Proposition 2

If $X \sim \mathrm{N}(0,1)$ then $X^2 \sim \mathrm{G}\left(\frac{1}{2}, \frac{1}{2}\right)$

Gamma distr. $G(\alpha, \nu)$

$$f_G(x) = \frac{1}{\Gamma(k)} \alpha^{\nu} e^{-\alpha x} x^{1-\nu}$$

proof.

$$\Pr[X^2 \le y] = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t = 2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t$$

Let $t^2 = s$ then $2t\mathrm{d}t = \mathrm{d}s$.

Thus,

$$\Pr[X^2 \le y] = 2\int_0^{y} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s}{2}\right) \frac{1}{2\sqrt{s}}\, \mathrm{d}s$$

$$= \int_0^{y} \frac{1}{\sqrt{\pi}} \left(\frac{1}{2}\right)^{\frac{1}{2}} \mathrm{e}^{-\frac{s}{2}}\, s^{\frac{1}{2}-1}\, \mathrm{d}s$$

Exponential distr., Gamma distr., and Poisson distr.

Question 3.

Suppose $X_1 \sim G(\alpha, \nu_1)$ and $X_2 \sim G(\alpha, \nu_2)$ are independent, then $X := X_1 + X_2 \sim$?

consider the case that ν_1 and ν_2 are integers.

Exponential distr., Gamma distr., and Poisson distr.

Proposition 3

Suppose $X_1 \sim G(\alpha, \nu_1)$ and $X_2 \sim G(\alpha, \nu_2)$ are independent, then $X := X_1 + X_2 \sim G(\alpha, \nu_1 + \nu_2)$

consider the case that ν_1 and ν_2 are integers.

Exponential distr., Gamma distr., and Poisson distr.

Proposition 3

Suppose $X_1 \sim G(\alpha, \nu_1)$ and $X_2 \sim G(\alpha, \nu_2)$ are independent,
then $X := X_1 + X_2 \sim G(\alpha, \nu_1 + \nu_2)$

consider the case that ν_1 and ν_2 are integers.

proof.

$$f_X(x) = \int_0^x f_{X_1}(t) f_{X_2}(x-t)\,\mathrm{d}t$$

$$= \int_0^x \frac{1}{\Gamma(\nu_1)} \alpha^{\nu_1} t^{\nu_1 - 1} \mathrm{e}^{-\alpha t} \frac{1}{\Gamma(\nu_2)} \alpha^{\nu_2} (x-t)^{\nu_2 - 1} \mathrm{e}^{-\alpha(x-t)}\,\mathrm{d}t$$

$$= \int_0^x \frac{\Gamma(\nu_1 + \nu_2)}{\Gamma(\nu_1)\Gamma(\nu_1)} \frac{1}{\Gamma(\nu_1 + \nu_2)} \alpha^{\nu_1 + \nu_2} t^{\nu_1 - 1} (x-t)^{\nu_2 - 1} \mathrm{e}^{-\alpha x}\,\mathrm{d}t$$

$$= \frac{1}{\Gamma(\nu_1 + \nu_2)} \alpha^{\nu_1 + \nu_2} \mathrm{e}^{-\alpha x} \int_0^x \frac{\Gamma(\nu_1 + \nu_2)}{\Gamma(\nu_1)\Gamma(\nu_1)} t^{\nu_1 - 1} (x-t)^{\nu_2 - 1}\,\mathrm{d}t$$

Exponential distr., Gamma distr., and Poisson distr.

Proposition 3

Suppose $X_1 \sim G(\alpha, \nu_1)$ and $X_2 \sim G(\alpha, \nu_2)$ are independent,

then $X := X_1 + X_2 \sim G(\alpha, \nu_1 + \nu_2)$

Let $s = \frac{t}{x}$, i.e., $t = xs$, then $\mathrm{d}t = x\mathrm{d}s$. Thus,

$$\int_0^x \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} t^{\nu_1-1}(x-t)^{\nu_2-1}\mathrm{d}t$$

$$= \int_0^1 \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} (xs)^{\nu_1-1}(x-xs)^{\nu_2-1} x\mathrm{d}s$$

$$= x^{\nu_1+\nu_2-1} \int_0^1 \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} s^{\nu_1-1}(1-s)^{\nu_2-1}\mathrm{d}s$$

$$= x^{\nu_1+\nu_2-1} \underline{\hspace{10em}}$$

Beta distr.

Thus, we obtain the claim. ■

Square of standard normal random variable

Proposition 4.

$X_1,\ldots,X_n \sim N(0,1)$, independently.

Let $Z := X_1^2 + \cdots + X_n^2$, then Z follows G(1/2,n/2).

χ^2 distribution with n degrees of freedom

By Propositions 2,3.

Proposition 2

If $X \sim N(0,1)$ then $X^2 \sim G(1/2,1/2)$

Proposition 3

Suppose $X_1 \sim G(\alpha,\nu_1)$ and $X_2 \sim G(\alpha,\nu_2)$ are independent,

then $X := X_1+X_2 \sim G(\alpha,\nu_1+\nu_2)$