共通基礎科目

確率統計・特論 (Probability and Statistics)

Todays topics

- what is probability?
- probability space

来嶋 秀治 (Shuji Kijima)

システム情報科学研究院 情報学部門

Dept. Informatics, ISEE

Preview

About this class

Evaluation

supplemental option for credit
not mandatory, but recommend.

- exercises (レポート課題) =: z (/10)
  - Almost every class (10回程度)
  - Submit hard copy at the end of class, or

    email to kijima@inf.kyushu-u.ac.jp before then.
- midterm (中間試験): (early) Nov., =:x (/50)
- final (期末試験): late Jan or early Feb, =:y (/50)
- score = x+y (+ z), something like that...
- passing mark = 60. No makeup exam (追試は無い)

Lecture

as much as possible... sorry.

- Use slides and/or board.
- Language: speak in Japanese, write in English.
- mathematical backgrounds (undergraduate level)

  e.g., calculus (微積分), linear algebra (線形代数), etc...
- Questions are welcome in English or Japanese. Do not hesitate!
- Check the web page!

  http://tcslab.csce.kyushu-u.ac.jp/~kijima/

  you can find exercise, keywords, and schedule.
- Office hour: Fri(金曜日) 16:40–18:10 (W2, 7F, 741)
- Topics ⇒ see http://tcslab.csce.kyushu-u.ac.jp/~kijima/

References

- 藤澤洋徳, 確率と統計, 現代基礎数学 13, 朝倉書店, 2006, 4000円.
- M. Mitzenmacher, E. Upfel,

Probability and Computing: Randomized Algorithms and Probabilistic Analysis,
Cambridge University Press, 2005.
(邦訳) 小柴健史, 河内亮周, 確率と計算 --乱択アルゴリズムと確率的解析--,
共立出版, 2009.

[確率論]

- 伏見正則, 確率と確率過程, 朝倉書店, 2002, 2800円

[統計学]

- 日本統計学会, 統計学 (統計検定1級対応), 東京図書, 2013.
- 東大教養学部統計学教室編, 統計学入門, 東京大学出版会, 1991, 2940円.

Check the web page!

http://www.tcslab.csce.kyushu-u.ac.jp/~kijima/

Requirement (前提知識)

Analysis (解析学)

- Differentiation (微分)
  - $\frac{\mathrm{d}}{\mathrm{d}x}\log(x)$?
  - Integration by parts (部分積分)
- $\int_a^b x\exp(-x)\,\mathrm{d}x$

Linear algebra (線形代数学)

- Bases (基底)
- Eigenvalues/diagonalization (固有空間/対角化)

What is Probability?

Examples

Ex 1. Monty Hall problem --- ask Marilyn

- ✓ You are given the choice of three doors:
  - ➢ Behind on door is a car; behind the others goats.
- ✓ You pick a door, say "A".
- ✓ The host (Monty), who knows what's behind the doors, opens another door, say "C", which he knows has a goat.
- ✓ He then says to you, "Do you want to pick door "B"?"

Question

Is it to your advantage to switch your choice?

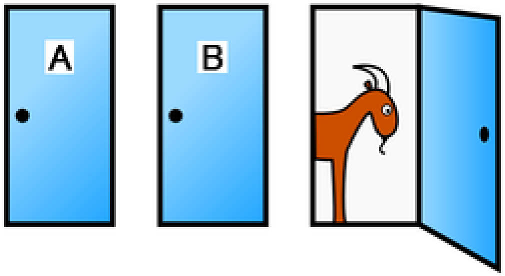


図: wikipedia"モンティーホール問題"より

Ex 1. A modified Monty Hall problem --- ask Marilyn

- ✓ You are given the choice of 26 doors:
  - ➢ Behind on door is a car; behind the others goats.
- ✓ You pick a door, say “A”.
- ✓ The host (Monty), who knows what's behind the doors, opens all other doors, say “C” to “Z”, except for “B”, each of which he knows has a goat.
- ✓ He then says to you, "Do you want to pick door “B”?”

Question

Is it to your advantage to switch your choice?

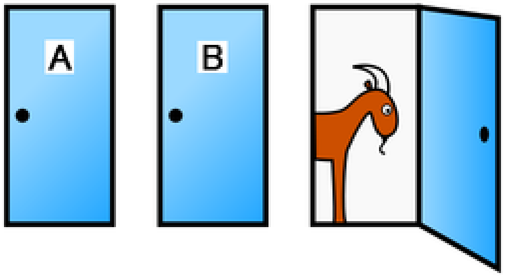


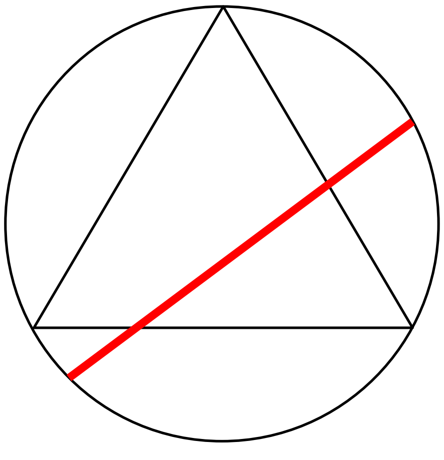
図: wikipedia”モンティーホール問題”より

Ex 2. Bertrand paradox

- ✓ Consider an equilateral triangle inscribed in a circle.
- ✓ Suppose a chord of the circle is chosen at random.

Question

What is the probability that the chord is longer than a side of the triangle?



Ex 2. Bertrand paradox

- ✓ Consider an equilateral triangle inscribed in a circle.
- ✓ Suppose a chord of the circle is chosen at random.

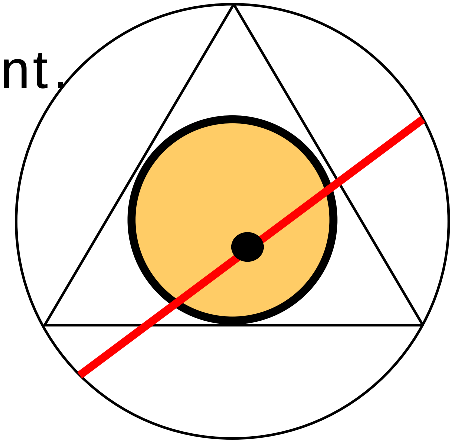
Question

What is the probability that the chord is longer than a side of the triangle (=:x)?

Answer 1: The "random midpoint" method.

- ✓ Choose a point anywhere within the circle, and construct a chord with the chosen point as its midpoint.
- ✓ The chord is longer than x iff the chosen point within a small circle.

$\Rightarrow$ the probability is 1/4



Ex 2. Bertrand paradox

- ✓ Consider an equilateral triangle inscribed in a circle.
- ✓ Suppose a chord of the circle is chosen at random.

Question

What is the probability that the chord is longer than a side of the triangle (=:x)?

Answer 2: The "random radius" method:

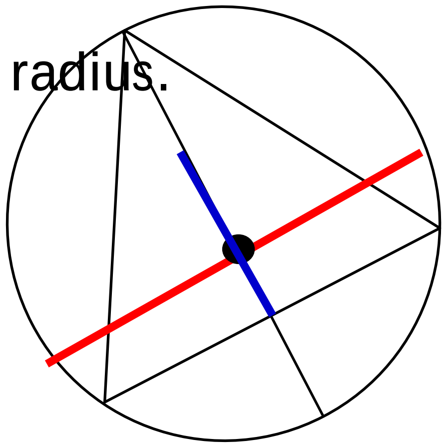
✓ Choose a radius of the circle and a point on the radius.

the chord through this point and perpendicular to the radius.

✓ The chord is longer than x

iff the chosen point on a blue line.

$\Rightarrow$ the probability is 1/2



Ex 2. Bertrand paradox

- ✓ Consider an equilateral triangle inscribed in a circle.
- ✓ Suppose a chord of the circle is chosen at random.

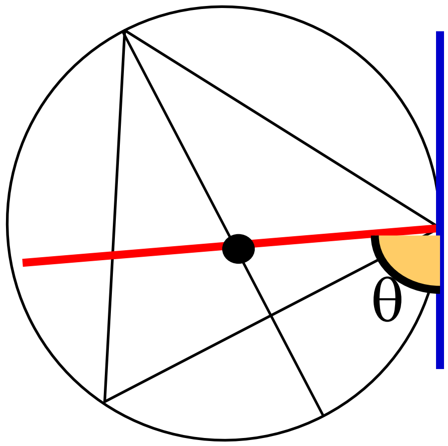
Question

What is the probability that the chord is longer than a side of the triangle (=:x)?

Answer 3: The "random endpoints" method::

- ✓ Choose two random points on the circumference of the circle and draw the chord joining them.
- ✓ The chord is longer than x iff $\pi/3 \leq \theta \leq 2\pi/3$

$\Rightarrow$ the probability is 1/3



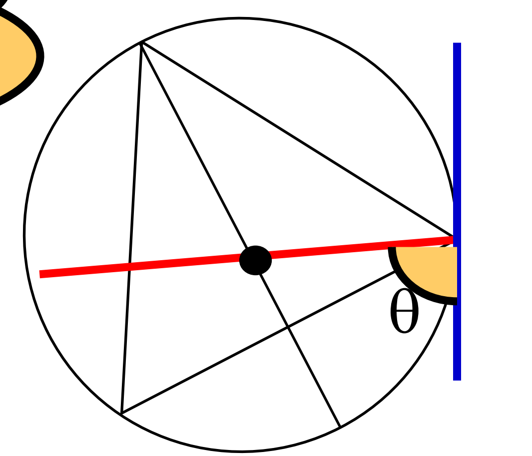
Ex 2. Bertrand paradox

- ✓ Consider an equilateral triangle inscribed in a circle.
- ✓ Suppose a chord of the circle is chosen at random.

Question

What is the probability that the chord is longer than a side of the triangle (=:x)?

What does
“a chord of the circle is chosen at random”
mean?



Probability Space

- ✓ Definitions
- ✓ Axiom
- ✓ Terminology

Definition: Probability Space

A probability space is defined by $(\Omega, \mathcal{F}, P)$

Ω: sample space(標本空間); a set of elementally events(標本点),

- an event(事象) is a subset of Ω.

$\mathcal{F}$: σ-algebra ($\subseteq 2^{\Omega}$); a set of events.

P: probability measure(確率測度); a function $\mathcal{F} \to \mathbb{R}$,

- probability of an event.

$\mathcal{F}$ is a σ-algebra

1. $\mathcal{F}$ contains the empty set: $\emptyset \in \mathcal{F}$
2. $\mathcal{F}$ is closed under complements: $A \in \mathcal{F} \Rightarrow \overline{A} \in \mathcal{F}$
3. $\mathcal{F}$ is closed under countable unions: $A_i \in \mathcal{F} \Rightarrow \bigcup_i A_i \in \mathcal{F}$

Definition: Probability Space

A probability space is defined by $(\Omega, \mathcal{F}, P)$

Ω: sample space(標本空間); a set of elementally events(標本点),

- an event(事象) is a subset of Ω.

$\mathcal{F}$: σ-algebra ($\subseteq 2^{\Omega}$); a set of events.

P: probability measure(確率測度); a function $\mathcal{F} \to \mathbb{R}$,

- probability of an event.

P is a probability measure (Kolmogorov axioms)

1. $P(A) \geq 0$ for any $A \in \mathcal{F}$.
2. $P(\Omega) = 1$.
3. Any countable sequence of mutual exclusive events $A_1, A_2, \ldots$ satisfies that $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$.

Example: Probability Space

A probability space is defined by $(\Omega, \mathcal{F}, P)$

Ω: sample space(標本空間); a set of elementally events(標本点),

- an event(事象) is a subset of Ω.

$\mathcal{F}$: σ-algebra ($\subseteq 2^{\Omega}$); a set of events.

P: probability measure(確率測度); a function $\mathcal{F} \to \mathbb{R}$,

- probability of an event.

Ex. 1. Die

- ✓ $\Omega = \{1,2,3,4,5,6\}$,
- ✓ $\mathcal{F} = 2^{\Omega}$
- ✓ $P(A) = \frac{|A|}{6}$ for any $A \subseteq \Omega$.

Example: Probability Space

A probability space is defined by $(\Omega, \mathcal{F}, P)$

Ω: sample space(標本空間); a set of elementally events(標本点),

- ➢ an event(事象) is a subset of Ω.

$\mathcal{F}$: σ-algebra ($\subseteq 2^{\Omega}$); a set of events.

P: probability measure(確率測度); a function $\mathcal{F} \rightarrow \mathbb{R}$,

- ➢ probability of an event.

Ex. 2. product of a pair of dice cast

Suppose we can only observe the parity of the product x·y of a pair of casts (x,y).

- ✓ Ω={(1,1), (1,2), (1,3), … , (4,5), (4,6), (5,6)},
- ✓ F={$\emptyset$, even, odd, Ω},
- ✓ P($\emptyset$)=0, P(even)=?, P(odd)=? , P(Ω)=1. => exercise

Property of Probability (1/2)

Kolmogolov's axiom (P1), (P2), (P3) implies the following.

Thm. 1

Let $(\Omega, \mathcal{F}, P)$ be an arbitrary probability space.

(1) $P(\emptyset) = 0$.

(2) If $A \subseteq B$ $(A, B \in \mathcal{F})$, then $P(B \setminus A) = P(B) - P(A)$.

(rem. $B \setminus A = B \cap \bar{A}$)

(3) If $A \subseteq B$ $(A, B \in \mathcal{F})$, then $P(A) \leq P(B)$.

(4) $P(A) \leq 1$ for any $A \in \mathcal{F}$.

(5) $P(\bar{A}) = 1 - P(A)$ for any $A \in \mathcal{F}$.

Property of Probability (2/2)

Kolmogolov's axiom (P1), (P2), (P3) implies the following.

Thm. 1. (cont.)

(6) P $P(A \cup \mathrm{B}) = P(A) + P(B) - P(A \cap B)$ for any $A \in \mathcal{F}$.

(7) If $A_1, A_2, \ldots \in \mathcal{F}$ satisfy $A_1 \subseteq A_2 \subseteq \cdots$,

then $P(\bigcup_i A_i) = \lim_{n \to \infty} P(A_n)$

(8) If $A_1, A_2, \ldots \in \mathcal{F}$ satisfy $A_1 \supseteq A_2 \supseteq \cdots$,

then $P(\bigcap_i A_i) = \lim_{n \to \infty} P(A_n)$

(9) For any $A_1, A_2, \ldots \in \mathcal{F}$,

$$P(\bigcup_i A_i) = 0 \Leftrightarrow \forall i \geq 0, P(A_i) = 0$$